Dynamical pressure for fluid mixtures with several temperatures

Henri Gouin, Tommaso Ruggeri

Dynamical pressure for fluid mixtures with several temperatures

HENRI GOUIN[a], **TOMMASO RUGGERI**[b]

a.Aix-Marseille Université, CNRS, Centrale Marseille, M2P2 UMR 7340, 13451 Marseille, France
Email : henri.gouin@univ-amu.fr ; henri.gouin@yahoo.fr
b. Department of Mathematics and Research Center of Applied Mathematics C.I.R.A.M. University of Bologna, Via Saragozza 8, 40123-I Bologna
Email : tommaso.ruggeri@unibo.it

Abstract :

We consider binary mixtures of fluids with components having different temperatures. A new dynamical pressure term is associated with the difference of temperatures between components even if fluid viscosities are null. The non-equilibrium dynamical pressure can be measured and may be convenient in several physical situations as for example in cosmological circumstances where a dynamical pressure played a major role in the evolution of the early universe.

Key words : *Fluid mixtures ; Multi-temperatures ; Dynamical pressure ; Hamilton's principle.*

Résumé :

Nous considérons des mélanges de fluides où chaque constituant a sa propre température. La différence de températures entre constituants implique l'existence d'une nouvelle pression dynamique même si les fluides ont une viscosité nulle. Cette pression dynamique peut être mesurée et utile dans de nombreuses situations physiques comme en cosmologie où une pression dynamique joue un rôle majeur dans l'évolution des débuts de l'univers.

Mots clés : *Mélanges de fluides ; multi-temperatures ; pression dynamique ; principe d'Hamilton.*

1 Introduction.

The theory of mixtures generally considers two different kinds of continua : homogeneous mixtures (each component occupies the whole mixture volume) and heterogeneous ones (each component occupies only a part of the mixture volume). At least four approaches to the construction of two-fluids models are known.
The first one for studying the heterogeneous two-phase flows is an averaging method [1]. A second approach was used for the construction of a quantum liquid model and was purposed for the homogeneous mixtures of fluids [2]. A third approach is done in the context of rational thermodynamics founded on the postulate that each constituent obeys the same balance laws as a single fluid [3, 4].
At least, there exists a different approach based on the Hamilton principle which is used for the construction of conservative mathematical models of continua. The variations of the Hamilton action are constructed in terms of virtual motions of continua which may be defined both in Lagrangian and Eulerian coordinates [5].
Here, we use variations in the case of fluid mixtures. The variational approach to the construction of two-fluid models has been used by many authors [6, 7, 8]. The method is different from the method proposed is [9] and is now developed in [10].
To study thermodynamical processes by using the Hamilton principle, the entropies of components are added to the field parameters instead of temperatures. The Lagrangian is the difference between the kinetic energy and an internal potential per unit volume depending on the densities, the entropies and the relative velocities of the mixture components. It is not necessary to distinguish molecular mixtures from heterogeneous fluids when each component occupies only a part of the mixture volume [11]. The terms including interaction between different components come from the direct knowledge of the internal potential.
The assumption of a common temperature for all the components is open to doubt for the suspensions of particles [12] as well as in the mixtures of gases in the early universe [13]. By using the Hamilton principle, the existence of several temperatures (one temperature for each component) must be associated with the existence of several entropies (one specific entropy for each component). The internal potential per unit volume is a function of the densities, the entropies and the difference of velocity between components.

2 Governing equations in conservative cases.

In a Galilean system of coordinates, the motion of a two-fluid mixture is represented as

$$\mathbf{Z}_\alpha = \mathbf{\Phi}_\alpha(\mathbf{z}), \qquad (\alpha = 1, 2)$$

where $\mathbf{z} = (t, \mathbf{x})$ denotes Eulerian coordinates in a four-dimensional domain ω in the time-space and $\mathbf{Z}_\alpha = (\lambda, \mathbf{X}_\alpha)$ denotes Lagrangian coordinates of the component α in a four-dimensional reference space ω_α. The conservation of matter for each component requires

$$\rho_\alpha \det \mathrm{F_a} = \rho_{\alpha 0}(\mathbf{X}_\alpha) \quad \text{with} \quad \mathrm{F_a} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}_\alpha}, \tag{1}$$

where index $_{\alpha 0}$ corresponds to the reference density in ω_α and $\det(\partial \mathbf{x}/\partial \mathbf{X}_\alpha)$ is the Jacobian determinant of the motion of the component α of density ρ_α. In differentiable cases Eqs (1) are equivalent to the equations of density balances

$$\frac{\partial \rho_\alpha}{\partial t} + \mathrm{div}(\rho_\alpha \mathbf{v}_\alpha) = 0, \tag{2}$$

where $\mathbf{v}_\alpha$ denotes the velocity of each component α. The Lagrangian of the binary system is

$$L = \sum_{\alpha=1}^{2} \left(\frac{1}{2}\, \rho_\alpha\, \mathbf{v}_\alpha^2 - \rho_\alpha \Omega_\alpha \right) - \eta(\rho_1, \rho_2, s_1, s_2, \mathbf{u}),$$

where the summation is taken over the fluid components $(\alpha = 1,\ 2)$, s_α are the specific entropies, $\mathbf{u} = \mathbf{v}_2 - \mathbf{v}_1$ is the relative velocity of components, Ω_α are the external force potentials, η is a potential per unit volume of the mixture. The Lagrangian L is a function of $\rho_\alpha, \mathbf{v}_\alpha, s_\alpha$ and we introduce the quantities

$$R_\alpha \equiv \frac{\partial L}{\partial \rho_\alpha} = \frac{1}{2}\, \mathbf{v}_\alpha^2 - \frac{\partial \eta}{\partial \rho_\alpha} - \Omega_\alpha, \quad \mathbf{k}_\alpha^T \equiv \frac{1}{\rho_\alpha} \frac{\partial L}{\partial \mathbf{v}_\alpha} = \mathbf{v}_\alpha^T - \frac{(-1)^\alpha}{\rho_\alpha} \frac{\partial \eta}{\partial \mathbf{u}}, \quad \rho_\alpha\, T_\alpha \equiv -\frac{\partial L}{\partial s_\alpha} = \frac{\partial \eta}{\partial s_\alpha}, \tag{3}$$

where ^{T} denotes the transposition and $\partial L/\partial \mathbf{v}_\alpha$, $\partial \eta/\partial \mathbf{u}$ are linear forms. Relation $(3)_3$ defines the temperatures T_α $(\alpha = 1, 2)$ which are dynamical quantities depending on ρ_1, ρ_2, s_1, s_2 and $\mathbf{u}$.

To obtain the equations of component motions by means of the Hamilton principle, we consider variations of particle motions in the form of surjective mappings $\mathbf{X}_\alpha = \mathbf{\Xi}_\alpha(t, \mathbf{x}; \kappa_\alpha)$, where scalars κ_α are defined in a neighborhood of zero ; they are associated with a two-parameter family of virtual motions. The real motions correspond to $\kappa_\alpha = 0$ such that $\mathbf{\Xi}_\alpha(t, \mathbf{x}; 0) = \phi_\alpha(t, \mathbf{x})$ and virtual Lagrange displacements are

$$\delta_\alpha \mathbf{X}_\alpha = \frac{\partial \mathbf{\Xi}_\alpha(t, \mathbf{x}; \kappa_\alpha)}{\partial \kappa_\alpha}\,|_{\kappa_\alpha = 0}\ , \quad (\alpha = 1, 2). \tag{4}$$

The Hamilton action is $a = \displaystyle\int_\omega L\, dv\, dt$. *We first consider the Hamilton principle in the form*

$$\delta_\alpha a \equiv \left(\frac{da}{d\kappa_\alpha} \right)_{|_{\kappa_\alpha = 0}} = \delta_\alpha \int_\omega L\, dv\, dt = 0,$$

under constraints (1), where $\delta_\alpha a$ are the variations of a associated with Rel. (4).
From the definition of virtual motions, we obtain the values of $\delta_\alpha \mathbf{v}_\alpha(\mathbf{x}, t)$, $\delta_\alpha \rho_\alpha(\mathbf{x}, t)$ and $\delta_\alpha s_\alpha(\mathbf{x}, t)$ where the notation $\delta_\alpha b(t, \mathbf{x})$ represents the variation of function b at $(t,\ \mathbf{x})$ fixed. The functions are assumed to be smooth enough in the domain ω_α and $\delta_\alpha \mathbf{X}_\alpha = 0$ on its boundary. It follows [10],

$$\delta_\alpha a = \int_{\omega_\alpha} \rho_{\alpha 0} \left(-\frac{\partial R_\alpha}{\partial \mathbf{X}_\alpha} + \frac{\partial}{\partial \lambda}(\mathbf{k}_\alpha^T\, F_\alpha) - T_\alpha \frac{\partial s_{\alpha 0}}{\partial \mathbf{X}_\alpha} \right) \delta_\alpha \mathbf{X}_\alpha\, dv_\alpha\, dt,$$

and we get the component motion equations in Lagrangian coordinates, $\dfrac{\partial}{\partial \lambda}(\mathbf{k}_\alpha^T\, F_\alpha) - \dfrac{\partial R_\alpha}{\partial \mathbf{X}_\alpha} - T_\alpha \dfrac{\partial s_{\alpha 0}}{\partial \mathbf{X}_\alpha} = 0.$

By taking into account the identity $\dfrac{d_\alpha F_\alpha}{dt} - \dfrac{\partial \mathbf{v}_\alpha}{\partial \mathbf{x}}\, F_\alpha = 0$ and for $\lambda = t$, we rewrite the equations in Eulerian coordinates,

$$\frac{d_\alpha \mathbf{k}_\alpha^T}{dt} + \mathbf{k}_\alpha^T \frac{\partial \mathbf{v}_\alpha}{\partial \mathbf{x}} = \frac{\partial R_\alpha}{\partial \mathbf{x}} + T_\alpha \frac{\partial s_\alpha}{\partial \mathbf{x}}\ .$$

To obtain the equation of energy, *we need a second variation of motions associated with the time parameter*. The variation corresponds to a virtual motion in the form $\lambda = \varphi(t;\kappa)$, where scalar κ is defined in a neighborhood of zero. The real motion of the mixture corresponds to $\kappa = 0$ such that $\varphi(t,0) = t$; the associated virtual displacement is

$$\delta\lambda = \frac{\partial\varphi(t;\kappa)}{\partial\kappa}\,|_{\kappa=0}\,.$$

In multi-component fluids, due to exchanges of energy between the components, the entropies cannot be conserved along component paths ; in the reference spaces ω_α, the specific entropies s_α depend also on λ

$$s_\alpha = s_{\alpha 0}\,(\lambda, \mathbf{X}_\alpha)\,.$$

The variation of Hamilton's action associated with the second family of virtual motions yields

$$\delta a \equiv\ \delta \int_\omega L\; dv\, dt\ =\ \int_\omega \frac{\partial L}{\partial\lambda}\,\delta\lambda\; dv\, dt\ = 0.$$

From $\dfrac{\partial L}{\partial\lambda} = \displaystyle\sum_{\alpha=1}^{2} \frac{\partial L}{\partial s_\alpha}\,\frac{\partial s_{\alpha 0}}{\partial\lambda}$, we deduce when $\lambda = t$, $\dfrac{\partial L}{\partial\lambda} = -\displaystyle\sum_{\alpha=1}^{2} \rho_\alpha T_\alpha \frac{d_\alpha s_\alpha}{dt}$, where $\dfrac{d_\alpha s_\alpha}{dt} = \dfrac{\partial s_\alpha}{\partial t} + \dfrac{\partial s_\alpha}{\partial \mathbf{x}}\,\mathbf{v}_\alpha$ is the material derivative with respect to velocity $\mathbf{v}_\alpha$. We obtain for the total mixture

$$\sum_{\alpha=1}^{2} \rho_\alpha T_\alpha \frac{d_\alpha s_\alpha}{dt} = 0. \tag{5}$$

Due to Eqs. (2) we obtain the equivalent form

$$\sum_{\alpha=1}^{2} Q_\alpha = 0 \qquad \text{with} \qquad Q_\alpha = \left(\frac{\partial \rho_\alpha s_\alpha}{\partial t} + \operatorname{div}(\rho_\alpha \mathrm{s}_\alpha \mathbf{v}_\alpha)\right) T_\alpha. \tag{6}$$

Equation (6) expresses that the exchange of energy between components has a null total amount. In case of mixtures with two entropies, the Hamilton principle is not able to close the system of motion equations ; we need additional arguments to obtain the evolution equations for each entropy s_α by considering the behaviors of Q_α. A possibility to close the system of equations is to consider that the momenta and heat exchanges between the components are rapid enough to have a common temperature. Another possibility, used by Landau for quantum fluids [2], is to assume that the total specific entropy s is convected along the first component trajectory. These assumptions are not valid for heterogeneous mixtures where each phase has different pressures and temperatures [12, 13].

3 Mixtures weakly out of equilibrium.

We consider the case when the mixture is *weakly out of equilibrium* such that the difference of velocities $\mathbf{u}$ and the difference of temperatures $T_2 - T_1$ are small enough with respect to the main field variables.

In the following, $\rho\,\mathbf{v} = \displaystyle\sum_{\alpha=1}^{2} \rho_\alpha \mathbf{k}_\alpha = \sum_{\alpha=1}^{2} \rho_\alpha \mathbf{v}_\alpha$ is the total momentum and $\rho = \displaystyle\sum_{\alpha=1}^{2} \rho_\alpha$ is the mixture density.

For the sake of simplicity, we neglect the external forces. Generally, the volume potential η is developed in the form [1]

$$\eta(\rho_1, \rho_2, s_1, s_2, \mathbf{u}) = e(\rho_1, \rho_2, s_1, s_2) - b(\rho_1, \rho_2, s_1, s_2)\,\mathbf{u}^2,$$

where b is a positive function of ρ_1, ρ_2, s_1, s_2. We consider the linear approximation when $|\mathbf{u}|$ is small with respect to $|\mathbf{v}_1|$ and $|\mathbf{v}_2|$. In linear approximation the volume potential is equal to the volume internal energy e [11],

$$\eta(\rho_1, \rho_2, s_1, s_2, \mathbf{u}) \approx e(\rho_1, \rho_2, s_1, s_2) = \rho\,\varepsilon(\rho_1, \rho_2, s_1, s_2),$$

where ε denotes the internal energy per unit mass. Let us note that the diffusion vector $\mathbf{j} = \rho_1(\mathbf{v}_1 - \mathbf{v}) \equiv \rho_2(\mathbf{v} - \mathbf{v}_2)$ is a small momentum vector deduced both from velocities and densities of the components. The equations of density balances (1) can be written in the form

$$\frac{d\rho}{dt} + \rho \operatorname{div}\mathbf{v} = 0 \qquad \text{and} \qquad \rho\,\frac{dc}{dt} + \operatorname{div}\mathbf{j} = 0, \tag{7}$$

1. In Ref. [9], the internal energy is the sum of the internal energies of the components $\left(\rho\,\varepsilon = \sum_{\alpha=1}^{2} \rho_\alpha \varepsilon_\alpha(\rho_\alpha, s_\alpha)\right)$.

where $c = \rho_1/\rho$ denotes the concentration of component 1 and $d/dt = \partial/\partial t + \partial/\partial \mathbf{x}\,.\,\mathbf{v}$ is the material derivative with respect to the average velocity of the mixture. The divergence of a linear operator $\mathbf{A}$ is the covector $\mathrm{div}\mathbf{A}$ such that, for any constant vector $\mathbf{a}$, $(\mathrm{div}\,\mathbf{A})\,\mathbf{a} = \mathrm{div}\,(\mathbf{A}\,\mathbf{a})$ and we write $\mathbf{v}_\alpha \mathbf{v}_\alpha^T \equiv \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha$. Let us denote by $h_\alpha \equiv \partial e/\partial \rho_\alpha$ the specific enthalpy of the component α. For processes with weak diffusion, the equations of component motions get the form,

$$\rho_\alpha\, \mathbf{\Gamma}_\alpha \equiv \frac{\partial \rho_\alpha \mathbf{v}_\alpha}{\partial t} + \mathrm{div}(\rho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha)^T = \rho_\alpha T_\alpha \,\mathrm{grad}\, s_\alpha - \rho_\alpha \,\mathrm{grad}\, h_\alpha.$$

The equation of total momentum is [10] : $$\frac{\partial \rho \mathbf{v}}{\partial t} + \mathrm{div}\left(\sum_{\alpha=1}^{2} (\rho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha) - \mathbf{t}\right)^T = 0,$$

where $\mathbf{t} = \sum_{\alpha=1}^{2} \mathbf{t}_\alpha$ is the total stress tensor such that $\mathbf{t}_{\alpha\nu\gamma} = -p_\alpha\ \delta_{\nu\gamma}$, $p_\alpha = \rho\rho_\alpha \varepsilon_{,\rho_\alpha} = \rho_\alpha e_{,\rho_\alpha} - \rho_\alpha e/\rho$, $p = \sum_{\alpha=1}^{2} p_\alpha$. The internal energy is a natural function of densities and entropies. Due to Def. $(3)_3$,

$$\rho_1\, T_1 = \rho\, \frac{\partial \varepsilon}{\partial s_1}(\rho_1, \rho_2, s_1, s_2) \quad \text{and} \quad \rho_2\, T_2 = \rho\, \frac{\partial \varepsilon}{\partial s_2}(\rho_1, \rho_2, s_1, s_2). \tag{8}$$

Let us denote by $\overline{\varepsilon}$ the expression of the specific internal energy as a function of ρ, c, s_1, s_2 such that $\overline{\varepsilon}(\rho, c, s_1, s_2) = \varepsilon(\rho_1, \rho_2, s_1, s_2)$; we get :

$$\rho\, \frac{d\varepsilon}{dt} = \rho\, \frac{\partial \overline{\varepsilon}}{\partial \rho}\, \frac{d\rho}{dt} + \rho\, \frac{\partial \overline{\varepsilon}}{\partial c}\, \frac{dc}{dt} + \rho\, \frac{\partial \overline{\varepsilon}}{\partial s_1}\, \frac{ds_1}{dt} + \rho\, \frac{\partial \overline{\varepsilon}}{\partial s_2}\, \frac{ds_2}{dt}.$$

Due to the fact that $\rho^2\, \dfrac{\partial \overline{\varepsilon}}{\partial \rho} = p$ and $\dfrac{\partial \overline{\varepsilon}}{\partial c} = h_1 - h_2$, we obtain

$$\rho\, \frac{d\varepsilon}{dt} = \frac{p}{\rho}\, \frac{d\rho}{dt} + \rho\,(h_1 - h_2)\, \frac{dc}{dt} + \rho_1\, T_1\, \frac{ds_1}{dt} + \rho_2\, T_2\, \frac{ds_2}{dt}. \tag{9}$$

By taking into account that $\dfrac{d_\alpha s_\alpha}{dt} = \dfrac{ds_\alpha}{dt} + \dfrac{\partial s_\alpha}{\partial \mathbf{x}}(\mathbf{v}_\alpha - \mathbf{v})$ and by using Eqs. (5), (7), Eq. (9) yields

$$\rho\, \frac{d\varepsilon}{dt} + p\, \mathrm{div}\, \mathbf{v} + (h_1 - h_2)\, \mathrm{div}\, \mathbf{j} + (T_1\, \mathrm{grad}\, s_1 - \mathrm{T}_2\, \mathrm{grad}\, s_2)^T \mathbf{j} = 0. \tag{10}$$

Due to Eqs. (8), the internal energy can be expressed as a function of densities and temperatures of components

$$\tilde{\varepsilon}(\rho_1, \rho_2, T_1, T_2) = \varepsilon(\rho_1, \rho_2, s_1, s_2).$$

As we wrote in [9], we define the average temperature T associated with T_1 and T_2 through the implicit solution of the equation

$$\tilde{\varepsilon}(\rho_1, \rho_2, T, T) = \tilde{\varepsilon}(\rho_1, \rho_2, T_1, T_2). \tag{11}$$

We denote by $\Theta_\alpha = T_\alpha - T$ the difference between component and average temperatures, which are non-equilibrium thermodynamical variables. Near equilibrium, Eq. (11) can be expanded to the first order ; then

$$\sum_{\alpha=1}^{2} c_v^\alpha\, \Theta_\alpha = 0 \qquad \text{with} \qquad c_v^\alpha = \frac{\partial \widetilde{\varepsilon}}{\partial T_\alpha}(\rho_1, \rho_2, T, T). \tag{12}$$

Due to the fact that $\rho\, d\varepsilon = \sum_{\alpha=1}^{2} \rho_\alpha\, T_\alpha\, ds_\alpha + \dfrac{p_\alpha}{\rho_\alpha}\, d\rho_\alpha$, then

$$\rho\, c_v^1 = T \sum_{\alpha=1}^{2} \rho_\alpha\, \frac{\partial s_\alpha}{\partial T_1}(\rho_1, \rho_2, T, T) \qquad \text{and} \qquad \rho\, c_v^2 = T \sum_{\alpha=1}^{2} \rho_\alpha\, \frac{\partial s_\alpha}{\partial T_2}(\rho_1, \rho_2, T, T). \tag{13}$$

The definition of the total entropy s of the mixture is

$$\rho\, s = \sum_{\alpha=1}^{2} \rho_\alpha s_\alpha(\rho_1, \rho_2, T_1, T_2). \tag{14}$$

The first order expansion of Eq. (14) yields

$$\rho\, s = \sum_{\alpha=1}^{2} \rho_\alpha s_\alpha(\rho_1,\rho_2,T,T) + \rho_\alpha s_\alpha \frac{\partial s_\alpha}{\partial T_1}(\rho_1,\rho_2,T,T)\,\Theta_1 + \rho_\alpha s_\alpha \frac{\partial s_\alpha}{\partial T_2}(\rho_1,\rho_2,T,T)\,\Theta_2.$$

Due to Rels. (12), (13), $\rho\, s = \sum_{\alpha=1}^{2} \rho_\alpha s_\alpha(\rho_1,\rho_2,T,T)$ and the specific entropy s does not depend on Θ_1 and Θ_2 but only on ρ_1, ρ_2 and T. We denote by $\hat{\varepsilon}$ the internal specific energy as a function of ρ, c, T :

$$\hat{\varepsilon}(\rho, c, T) = \tilde{\varepsilon}(\rho_1, \rho_2, T, T),$$

which satisfies the Gibbs equation

$$T ds = d\hat{\varepsilon} - \frac{p_o}{\rho^2} d\rho + (\mu_2 - \mu_1)\, dc$$

where $p_o\,(\rho, c, T)$ is the equilibrium pressure at temperature T and $\mu_2 - \mu_1$, difference of component chemical potentials, is the chemical potential of the whole mixture. By taking into account of equation (7), we get

$$\rho \frac{d\hat{\varepsilon}}{dt} + p_o \,\mathrm{div}\,\mathbf{v} + (\mu_1 - \mu_2)\ \mathrm{div}\,\mathbf{j} - \rho T \frac{ds}{dt} = 0.$$

Moreover,

$$\rho \frac{ds}{dt} = \sum_{\alpha=1}^{2} \rho_\alpha \frac{d_\alpha s_\alpha}{dt} + \mathrm{div}\,[(s_2 - s_1)\mathbf{j}\,]. \tag{15}$$

Equation (15) yields the relation between the material derivatives of entropy s and entropies s_1 and s_2. By taking into account this result in Eq. (10) and $\hat{\varepsilon}(\rho, c, T) = \varepsilon(\rho_1, \rho_2, s_1, s_2)$, we obtain

$$T \sum_{\alpha=1}^{2} \rho_\alpha \frac{d_\alpha s_\alpha}{dt} + (p - p_o)\ \mathrm{div}\,\mathbf{v} + \Big((h_1 - h_2) - (\mu_1 - \mu_2) + T\,(s_2 - s_1)\Big)\,\mathrm{div}\,\mathbf{j} + (\Theta_1 \mathrm{grad} s_1 - \Theta_2 \mathrm{grad} s_2)^{\mathrm{T}} \mathbf{j} = 0. \tag{16}$$

The differences of temperatures $\Theta_1 \equiv T_1 - T$ and $\Theta_2 \equiv T_2 - T$ are small with respect to T and $\mathbf{j}$ is a small diffusion term with respect to the mixture momentum $\rho\mathbf{v}$; consequently, in an approximation to the first order, the term

$$(\Theta_1\,\mathrm{grad}\, s_1 - \Theta_2\,\mathrm{grad}\, s_2)^T\,\mathbf{j}$$

is negligible. Let us consider

$$K \equiv \Big((h_1 - h_2) - (\mu_1 - \mu_2) + T\,(s_2 - s_1)\Big)\,\mathrm{div}\,\mathbf{j}\,;$$

we get

$$K = \Big((h_1 - T_1 s_1) - (h_2 - T_2 s_2) - (\mu_1 - \mu_2) + \Theta_1 s_1 + \Theta_2 s_2\Big)\,\mathrm{div}\,\mathbf{j}\,.$$

In an approximation to the first order, the term $(\Theta_1 s_1 + \Theta_2 s_2)\,\mathrm{div}\,\mathbf{j}$ is negligible.

Due to the fact that $\mu_\alpha(\rho_1, \rho_2, T_1, T_2) = h_\alpha - T_\alpha s_\alpha$ is the chemical potential of the component α, when $\mathbf{j}$ is a small diffusion velocity with respect to average velocity $\mathbf{v}$, the term

$$\Big(\mu_1(\rho_1,\rho_2,T_1,T_2) - \mu_2(\rho_1,\rho_2,T_1,T_2) - \Big(\mu_1(\rho_1,\rho_2,T) - \mu_2(\rho_1,\rho_2,T)\Big)\Big)\,\mathrm{div}\,\mathbf{j}$$

is vanishing in an approximation to the first order.
Consequently, in an approximation to the first order, Eq. (16) reduces to

$$\sum_{\alpha=1}^{2} \rho_\alpha \frac{d_\alpha s_\alpha}{dt} = -\frac{1}{T}\,(p - p_o)\ \mathrm{div}\,\mathbf{v}. \tag{17}$$

The exchange of energy between components must obey the second law of thermodynamics : the total entropy rate is an increasing function of time and we consider the second law of thermodynamics in the form

$$\sum_{\alpha=1}^{2}\left(\frac{\partial \rho_\alpha s_\alpha}{\partial t}+\mathrm{div}(\rho_\alpha \mathrm{s}_\alpha \mathbf{v}_\alpha)\right)\ \geq 0 \tag{18}$$

Due to Rels. (2) the Clausius-Duhem inequality (18) is equivalent to

$$\sum_{\alpha=1}^{2}\rho_\alpha\,\frac{d_\alpha s_\alpha}{dt}\ \geq 0\,.$$

This implies that the second member of Rel. (17) must be positive. Therefore, as usual in thermodynamics of irreversible processes, the entropy inequality requires

$$\pi \equiv p-p_o=-\Lambda\ \mathrm{div}\,\mathbf{v}\,. \tag{19}$$

This expression defines the Lagrange multiplier Λ of proportionality such that $\Lambda \geq 0$. The dynamical pressure π is the difference between the pressure in the process out of equilibrium with different temperatures for the components and the pressure of the mixture assumed in local thermodynamical equilibrium with the common average temperature T. Let us notice that Eqs. (6,19) allow to obtain Q_α values. In fact,

$$\rho_1 T\,(T_2-T_1)\frac{d_1 s_1}{dt}=\Lambda\,T_2\,(\mathrm{div}\,\mathbf{v})^2 \quad \text{and} \quad \rho_2 T\,(T_1-T_2)\frac{\mathrm{d}_2 s_2}{\mathrm{dt}}=\Lambda\,T_1\,(\mathrm{div}\,\mathbf{v})^2$$

and the system of field equations is now closed.

4 Conclusion.

The Hamilton principle points out that a dynamical pressure can be obtained by neglecting viscosity, friction or external heat fluxes. This is a main property of mixtures with multi-temperatures and this fact may have some applications in plasma of gases and in the evolution of the early universe [14].
The results are in complete accordance with the ones by Gouin & Ruggeri [9] and developed in [10]. This is an important verification of the fact that the Hamilton principle can be extended to nonconservative mixture motions when components have different temperature. A difference with classical thermodynamics methods is that the volume internal energy is not necessary the sum of the volume internal energies of the components. In [10], the volume internal energy is a nonseparate function of densities and entropies (or temperatures) and is consequently more general than in [9].

Références

[1] Ishii M., *Thermo-fluid dynamic theory of two-phase flows*, Paris, Eyrolles, 1975.

[2] Putterman S., *Super fluid hydrodynamics*, Elsevier, New York, 1974.

[3] Müller I., Ruggeri T., *Rational Extended Thermodynamics*, Springer-Verlag, New York, 1998.

[4] Ruggeri T., Simić S. , On the hyperbolic system of a mixture of Eulerian fluids : a comparison between single and multi-temperature models, *Math. Meth. Appl. Sci.* **30** (2007) 827-849.

[5] Serrin J., *Mathematical principles of classical fluid mechanics in* Encyclopedia of Physics VIII/1, Flügge Ed., Springer, Berlin, 1960.

[6] Berdichevsky V.L., *Variational principles of continuum mechanics,* Nauka, Moscow, 1983.

[7] Gouin H., Variational theory of mixtures in continuum mechanics, *Eur. J. Mech. B/Fluids* **9** (1990) 469-471.

[8] Gouin H., Ruggeri T., Mixture of fluids involving entropy gradients and acceleration waves in interfacial layers, *Eur. J. Mech. B/Fluids* **24** *(2005) 596-613.*

[9] Gouin H., Ruggeri T., Identification of an average temperature and a dynamical pressure in multi-temperature mixture of fluids, *Phys. Review E* **78** (2008) 016303.

[10] Gouin H., Ruggeri T., The Hamilton principle for fluid binary mixtures with two temperatures, *Boll. Unione Mat. Ital.*, **9**, II (2009).

[11] Gavrilyuk S.L., Gouin H., Perepechko Yu. V., Hyperbolic models of homogeneous two-fluid mixtures, *Meccanica* **33** (1998) 161-175.

[12] Lhuillier D., From Molecular mixtures to suspensions of particles, *J. Physique II* **5** (1995) 19-36.

[13] de Groot S.R., van Leeuwen W.A., van Weert Ch.G., *Relativistic kinetic theory*. North-Holland, Amsterdam (1980).

[14] Weinberg S., Entropy generation and the survival of protogalaxies in an expanding universe, *Astrophys. J.* **168**, (1971) 175-194.